

A NOTE ON PRODUCTS OF SQUARE-ZERO MATRICES OVER DIVISION RINGS

Vo Thi Van Anh

Ho Chi Minh City University of Technology and Engineering

Email: anhvtv@hcmute.edu.vn

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ABSTRACT

Let D be a division ring and let $n \geq 2$. We study factorizations of matrices in $M_n(D)$ into products of square-zero matrices with prescribed ranks. We show that a matrix $A \in M_n(D)$ is a product of square-zero matrices if and only if $\text{rank}(A) \leq n/2$. Moreover, under this condition, such factorizations can be refined to satisfy prescribed rank constraints on the factors. In particular, we prove that A admits a factorization into a product of six square-zero matrices whose ranks lie in a prescribed interval.

Keywords: *Division ring; matrix decomposition; nilpotent matrix; square-zero matrix*

1. Introduction

Let D be a division ring and let $M_n(D)$ denote the algebra of all $n \times n$ matrices over D . A matrix $A \in M_n(D)$ is called *nilpotent of index k* if

$$A^k = 0 \quad \text{and} \quad A^{k-1} \neq 0.$$

A matrix $A \in M_n(D)$ is called *square-zero* if it is nilpotent of index 2, equivalently,

$$A^2 = 0 \quad \text{and} \quad A \neq 0.$$

Factorizations of matrices into products of nilpotent or square-zero matrices have been extensively studied in matrix theory over fields and division rings. Classical results of Wu [1], Laffey [2] and Sourour [3] characterize products of nilpotent matrices over fields, while later works of Botha [4, 5], Bukovšek [6], Mohammadian [7] investigated products of square-zero matrices and commuting square-zero matrices under various assumptions and rank constraints.

In particular, Botha proved (see [5, Theorem 5]) that a matrix $A \in M_n(F)$ over a field F can be written as a product of square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{2}.$$

Moreover, every such matrix can be expressed as a product of at most three square-zero matrices with arbitrary r_i satisfying

$$\text{rank}(A) \leq r_i \leq \frac{n}{2}, i = 1, 2, 3.$$

In general, the number three cannot be reduced.

This result naturally motivates the study of square-zero factorizations over division rings, where rank is understood as the dimension of the right vector space generated by the columns of a matrix. In the noncommutative setting, additional difficulties arise from the lack of commutativity and the more delicate behavior of matrix decompositions.

In this paper, we study factorizations of matrices in $M_n(D)$ into products of square-zero matrices with prescribed ranks. We show that a matrix $A \in M_n(D)$ is a product of square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{2}.$$

Furthermore, we prove that such factorizations can be refined so that the square-zero factors have ranks belonging to prescribed intervals. In particular, every matrix satisfying

$$\text{rank}(A) \leq \frac{n}{2}$$

admits a factorization into a bounded number of square-zero matrices whose ranks satisfy prescribed constraints.

2. Literature Review

Factorizations of matrices into products of nilpotent matrices have been studied for several decades. One of the earliest results in this direction was obtained by Wu [1] in 1987, who proved that every singular complex matrix that is not a nonzero nilpotent matrix of size 2×2 can be expressed as a product of two nilpotent matrices whose ranks are both equal to the rank of the matrix.

Later, Laffey [2] and Sourour [3] extended this result to arbitrary fields by proving that every singular matrix over an arbitrary field, except for nonzero nilpotent matrices of size 2×2 , can be expressed as a product of two nilpotent matrices whose ranks are equal to the rank of the matrix.

In 1994, Botha [4] investigated factorizations into nilpotent matrices with prescribed ranks. He showed that a singular matrix $A \in M_n(F)$ can be expressed as a product of two nilpotent matrices with prescribed ranks r_1 and r_2 if and only if

$$\begin{aligned} \text{rank}(A) &\leq \min\{r_1, r_2\}, \\ \max\{r_1, r_2\} &\leq n - 1, \end{aligned}$$

and

$$r_1 + r_2 \leq n + \text{rank}(A),$$

except when A is a nonzero nilpotent 2×2 matrix of rank 1.

The study of products of square-zero matrices has also attracted considerable attention. In 2016, Botha [5] determined conditions under which a matrix over a field can be expressed as a product of square-zero matrices. Moreover, it is known that a matrix $A \in M_n(F)$ is a product of square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{2},$$

and every such matrix can be expressed as a product of at most three square-zero matrices.

Another direction concerns products of commuting square-zero matrices. In 2007, Bukovšek [6] proved that a matrix $A \in M_n(F)$ is a product of k pairwise commuting square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{2^k}.$$

In the same work, Bukovšek also characterized matrices that can be expressed as products of two commuting nilpotent matrices.

The corresponding problem over division rings was later studied by Mohammadian [7] in 2012. He proved that a matrix $A \in M_n(D)$ can be written as a product of two commuting square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{4}.$$

Although various results concerning products of nilpotent and square-zero matrices have been obtained over fields and division rings, the problem of characterizing products of square-zero matrices over arbitrary division rings has not been fully investigated. Moreover, factorizations with prescribed rank

conditions on the square-zero factors appear to remain unexplored.

Motivated by these questions, in this paper we study factorizations of matrices in $M_n(D)$ into products of square-zero matrices with prescribed ranks. We establish a characterization in terms of the rank of the matrix and obtain factorizations in which the square-zero factors satisfy prescribed rank constraints.

3. Theoretical Basis

Throughout, let D be a division ring and let n be a positive integer. We write $M_n(D)$ for the ring of all $n \times n$ matrices over D , and $GL_n(D)$ for the group of invertible matrices in $M_n(D)$.

For $\alpha \in D$, let $J_n(\alpha)$ denote the $n \times n$ Jordan block with diagonal entries α , and write

$$J_n := J_n(0).$$

Given matrices $A_i \in M_{m_i}(D)$ with $\sum_{i=1}^s m_i = n$, we write

$$\bigoplus_{i=1}^s A_i$$

for their block diagonal direct sum.

Recall that for a matrix $A \in M_{m \times n}(D)$, the rank of A , denoted by $\text{rank}(A)$, is defined as the dimension over D of the right subspace of D^m spanned by the columns of A .

It is well known that rank is invariant under multiplication by invertible matrices. More precisely, if $P \in GL_m(D)$ and $Q \in GL_n(D)$, then

$$\text{rank}(PAQ) = \text{rank}(A).$$

In particular, similar matrices over a division ring necessarily have the same rank.

We first recall the following characterization of products of square-zero matrices over fields obtained by Botha.

Theorem 1 ([5, Theorem 5]). *Let F be a field, and let $n \geq 2$ be an integer. Then $A \in M_n(F)$ is a product of square-zero matrices if and only if*

$$\text{rank}(A) \leq n/2.$$

Moreover, if $\text{rank}(A) \leq n/2$, then for any prescribed integers r_1, r_2, r_3 satisfying $\text{rank}(A) \leq r_i \leq n/2, i = 1, 2, 3$, there exist square-zero matrices A_1, A_2, A_3 such that $A = A_1 A_2 A_3$ and $\text{rank}(A_i) = r_i$.

We next recall several structural results on matrices over division rings that will be used throughout the paper.

The following canonical form for nilpotent matrices over division rings was obtained in [7, Theorem 5].

Lemma 2. *Let D be a division ring, and let n be a positive integer. Then every nilpotent matrix $N \in M_n(D)$ is similar to $\bigoplus_{i=1}^s J_{m_i}$, where $m_1 + m_2 + \cdots + m_s = n$.*

To deal with non-nilpotent singular matrices, we will use the following decomposition result, which follows from [8, Theorem 15, p. 28].

Lemma 3. *Let D be a division ring and let $n \geq 2$ be an integer. If $A \in M_n(D)$ is neither invertible nor nilpotent, then A is similar to $A_1 \oplus A_2$, where $A_1 \in M_t(D)$ is nilpotent for some $t \geq 1$, and $A_2 \in GL_{n-t}(D)$.*

The following factorization result is a consequence of [9, Theorem 2.1].

Lemma 4. *For any $A \in GL_n(D)$, there exist a lower triangular matrix L and an upper triangular matrix U such that A is similar to LU .*

The following elementary observation gives a necessary rank condition for square-zero matrices.

Lemma 5. *Let V be a right vector space over a division ring D , and let $\phi: V \rightarrow V$ be a right D -linear map such that $\phi^2 = 0$. Then $\dim_D \text{Im}(\phi) \leq 1/2 \dim_D V$. In particular, if $A \in M_n(D)$ satisfies $A^2 = 0$, then $\text{rank}(A) \leq n/2$.*

Proof. Since $\phi^2 = 0$, we have

$$\text{Im}(\phi) \subseteq \ker(\phi).$$

Thus

$$\begin{aligned} \dim_D V &= \dim_D \ker(\phi) + \dim_D \text{Im}(\phi) \\ &\geq 2 \dim_D \text{Im}(\phi). \end{aligned}$$

The matrix statement follows by considering the linear map $\phi(x) = Ax$. $\square$

4. Main Results

We now establish the main result of this paper, which characterizes matrices over division rings that can be expressed as products of square-zero matrices. We also show that such factorizations can be obtained with prescribed rank conditions on the factors.

Theorem 6. *Let D be a division ring, and let $n \geq 2$ be an integer. A matrix $A \in M_n(D)$ can be expressed as a product of square-zero matrices if and only if $\text{rank}(A) \leq n/2$.*

In this situation, one may choose a factorization of A into three square-zero matrices. If, in addition, the ranks of the factors are prescribed in advance, say $r_1, \dots, r_6$ satisfying

$$\text{rank}(A) \leq r_i \leq \frac{n}{2},$$

then A can be written as a product of six square-zero matrices A_i such that $\text{rank}(A_i) = r_i$.

Proof. First, assume that $A \in M_n(D)$ is a product of square-zero matrices. Then $\text{rank}(A)$ does not exceed the rank of each factor. By Lemma 5, every square-zero matrix has rank at most $n/2$. Hence

$$\text{rank}(A) \leq n/2.$$

Conversely, assume that $\text{rank}(A) = r$ with $r \leq n/2$. Let F be the center of D . We consider two cases.

Case 1. A is nilpotent. In this case, by Lemma 2, there exists a matrix $B \in M_n(F)$, where F is the center of D , such that B is similar to A . In particular, $\text{rank}(B) = \text{rank}(A) \leq n/2$. By Theorem 1, B is product of three square-zero matrices. Moreover, given integers $r_1, \dots, r_6$ satisfying

$$\text{rank}(B) \leq r_i \leq \frac{n}{2}, \quad i = 1, \dots, 6,$$

we can express B as a product

$$B = B_1 \cdots B_6,$$

where each B_i is a square-zero matrix with $\text{rank}(B_i) = r_i$, by iteratively applying Theorem 1.

Case 2. A is not nilpotent. By Lemma 3, there exist a nilpotent matrix $A_1 \in M_t(D)$, with $t \geq 1$, and a matrix $A_2 \in \text{GL}_{n-t}(D)$ such that A is similar to $A_1 \oplus A_2$. Then

$$\text{rank}(A_2) = n - t,$$

and hence

$$\begin{aligned} \text{rank}(A_1) &= r - \text{rank}(A_2) = r - n + t \\ &= \left(r - \frac{n}{2}\right) + \left(\frac{t}{2} - \frac{n}{2}\right) + \frac{t}{2} \leq \frac{t}{2}. \end{aligned}$$

By Lemma 2, A_1 is similar to $B \oplus 0_{t-m}$, where

$$B = \bigoplus_{i=1}^s J_{n_i} \in M_m(D), \quad n_i \geq 2.$$

Thus,

$$\text{rank}(A_1) = \text{rank}(B) = m - s,$$

and so

$$m - s \leq \frac{t}{2}.$$

It follows that A is similar to

$$(B \oplus 0_{m-2s}) \oplus (0_{t-2h} \oplus A_2),$$

where $h = m - s$. Denote

$$X = B \oplus 0_{m-2s},$$

$$Y = 0_{t-2h} \oplus A_2.$$

Then $X \in M_{2h}(F)$ and $\text{rank}(X) = h$. and

By Theorem 1, X is a product of three square-zero matrices. Since

$$(t - 2h) - (n - t) = 2 \left(\frac{n}{2} - \text{rank}(A) \right) \geq 0,$$

which allows us to define square-zero matrices Y_1, Y_2, Y_3 by

$$Y_1 = 0_{t-2h-(n-t)} \oplus \begin{pmatrix} 0_{n-t} & 0_{n-t} \\ I_{n-t} & 0_{n-t} \end{pmatrix},$$

$$Y_2 = 0_{t-2h-(n-t)} \oplus \begin{pmatrix} I_{n-t} & -I_{n-t} \\ I_{n-t} & -I_{n-t} \end{pmatrix},$$

and

$$Y_3 = 0_{t-2h-(n-t)} \oplus \begin{pmatrix} 0_{n-t} & A_2 \\ 0_{n-t} & 0_{n-t} \end{pmatrix},$$

we obtain $Y = Y_1 Y_2 Y_3$. Therefore, A is a product of three square-zero matrices.

Now suppose that integers $r_1, \dots, r_6$ are prescribed such that

$$\text{rank}(A) \leq r_i \leq n/2, \quad i = 1, \dots, 6.$$

Then there exist integers $\ell_1, \dots, \ell_6$ and $k_1, \dots, k_6$ such that

$$\begin{aligned} r_i &= \ell_i + k_i, \quad \text{rank}(A_1) \leq \ell_i \\ &\leq t/2, \quad \text{and} \quad k_i \\ &= \text{rank}(A_2) = n - t, \quad i \\ &= 1, \dots, 6. \end{aligned}$$

Since X is nilpotent, by Case 1 it can be written as a product of six square-zero matrices X_i with ranks ℓ_i . On the other hand, by Lemma 4, we may assume $A_2 = LU$. Then Y can be expressed as $Y = VH$, where

$$V = \begin{pmatrix} 0_{(t-2h) \times (n-t)} & 0_{(t-2h) \times (t-2h)} \\ L & 0_{(n-t) \times (t-2h)} \end{pmatrix},$$

$$H = \begin{pmatrix} 0_{(n-t) \times (t-2h)} & U \\ 0_{(t-2h) \times (t-2h)} & 0_{(t-2h) \times (n-t)} \end{pmatrix}.$$

Since $(t - 2h) - (n - t) \geq 0$, both V and H are nilpotent. It follows from Lemma 2 and Theorem 1 that Y is a product of six square-zero matrices with ranks k_i . Therefore, combining the decompositions of X and Y , we obtain that A is a product of six square-zero matrices with ranks $r_i = \ell_i + k_i$, for $i = 1, \dots, 6$. $\square$

5. Conclusion

In this paper, we investigated factorizations of matrices over division rings into products of square-zero matrices. We established that a matrix $A \in M_n(D)$ is a product of square-zero matrices if and only if

$$\text{rank}(A) \leq \frac{n}{2}.$$

We also obtained factorizations in which the square-zero factors satisfy prescribed rank constraints.

However, the optimality of the number of factors obtained in our constructions remains unclear. In particular, for prescribed ranks of the square-zero factors, it is still unknown whether the number of factors can always be reduced further. Determining optimal bounds for the minimal number of square-zero factors under prescribed rank conditions seems to be an interesting problem for future research.

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VỀ TÍCH CỦA CÁC MA TRẬN CÓ BÌNH PHƯƠNG BẰNG KHÔNG TRÊN VÀNH CHIA

Võ Thị Vân Anh

Trường Đại học Công nghệ Kỹ thuật Thành phố Hồ Chí Minh

Email: anhvtv@hcmute.edu.vn

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TÓM TẮT

Cho D là một vành chia và $n \geq 2$. Chúng tôi nghiên cứu các phân tích của những ma trận trong $M_n(D)$ thành tích của các ma trận có bình phương bằng không với hạng được chỉ định trước. Chúng tôi chỉ ra rằng một ma trận $A \in M_n(D)$ là tích của các ma trận có bình phương bằng không khi và chỉ khi $\text{rank}(A) \leq n/2$. Hơn nữa, dưới điều kiện này, ta có thể xây dựng các phân tích như vậy sao cho hạng của các nhân tử thỏa những điều kiện xác định. Đặc biệt, chúng tôi chứng minh rằng A có thể được phân tích thành tích của không quá sáu ma trận có bình phương bằng không, với hạng được chọn tùy ý trong một khoảng cho trước.

Từ khóa: *Vành chia; phân tích ma trận; ma trận lũy linh; ma trận có bình phương bằng không*